

# The reduced covariant phase space quantization of the three dimensional Nambu-Goto string

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## Abstract

The reduced covariant phase space associated with the three-dimensional Euclidean Nambu-Goto action can be identified, via the Enneper-Weierstrass representation of minimal surfaces, with the space of complex analytic functions plus three translational zero modes. The symplectic structure induced through the Enneper-Weierstrass map can be explicitly computed. Quantization is then straightforward, yielding as a result a target-space Euclidean-invariant, positive-definite, two-dimensional quantum field theory. The physical states are shown to correspond with particles states of integer spin and arbitrary mass.

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## 1 Introduction

String theory is certainly a fascinating topic. It aims towards a complete understanding of the principles of physical interaction solely in terms of geometry. Nevertheless, as could have been easily foreseen, such an ambitious program is plagued with almost unsurmountable difficulties. Although much progress has been achieved since the epic times of the first dual models of strong interactions, when the first hints were provided that a string model could be useful in understanding fundamental physical process, much still remains to be done. For example, the Nambu-Goto string model, the simplest and oldest among all of them, is still waiting to be consistently quantized outside the critical dimension. It is the purpose of this work to report on some progress in that direction.

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In the present work I will restrict myself to the three-dimensional case. The reasons to do so are manifold. First of all the theory of immersed surfaces is a subject that has been thoroughly studied since the pioneering days of Gauss, and therefore a plethora of deep geometrical results are directly available. This work will be directly based upon them; but this is certainly not the only reason. Three dimensions play in the case of strings the analogue of two dimensions in the particle case, i.e., the lowest dimension for which dynamics are not trivial in nature; it is therefore natural to expect it to be the simpler case. It is by now a futile exercise to try to justify the important role that two-dimensional quantum field theory has played in our current understanding of more realistic particle theories. It is my belief that three dimensional string theory must occupy a similar relevant place in the quantum theory of extended objects. Nevertheless it is clear that the importance of three-dimensional string theory transcends the limits of the so-called “theory of everything” and may find direct application in several, and perhaps more realistic, physical situations. The statistical mechanics of membranes, interfaces, and spin systems range among the most important among them. The transfer matrix formalism allows for a direct application of the quantum mechanics of three dimensional strings to this statistical mechanical models, and viceversa. In particular, it has been argued by Polyakov that the three-dimensional Ising model near criticality should be described by a three-dimensional string theory.

There have been traditionally two different approaches to the quantization of strings. The covariant approach, by far the most popular nowadays, is obtained by implementing the constraints coming from reparameterization invariance à la Dirac, or in more modern treatments via the construction of its associated BRST charge. It is well known that consistency of this approach requires the dimension of space-time to be 26. This fact induced Polyakov [8] to consider an alternative, and still covariant, line of attack based on the coupling of conformal matter to two-dimensional gravity. Although classically both approaches are easily seen to be equivalent for the case of  $D$  bosons coupled to two-dimensional gravity (where  $D$  represents the dimension of the target space), Polyakov’s treatment permitted, through a careful study of the Weyl anomaly, to extend the analysis to the non-critical case. Unfortunately, the existence of the infamous  $c = 1$  barrier has not yet allowed us to study the physically interesting dimensions, arguably 3 and 4.

The other main line of attack to the problem is the so-called light-cone gauge quantization. The idea behind it is to completely reduce the phase space of the string theory and quantize directly its physical transverse modes. There is an ongoing discussion in the literature about which is, on general principles, the “correct” way to quantize a dynamical system subject to constraints. There are many finite-dimensional examples for which the Dirac and the reduced phase space quantization algorithms yield different, but yet consistent, quantum theories. Nevertheless, it seems that physical input determines as

the correct algorithm one or the other, depending on the particular model, and that there is no particular one a priori preferred by nature. In the light-cone approach the reduced phase space is obtained through the choice of a non Poincaré covariant gauge condition. Roughly speaking, one of the coordinates in the world sheet is fixed to correspond to the time coordinate of the space-time in which the string lives. In this approach the magic number 26 pops out as the only dimension in which one can construct a representation of the Poincaré algebra. The technical reasons behind this come from the fact that due to the non-covariant gauge fixing the Poincaré group is nonlinearly realized at the field level, provoking ordering problems that seem only to be fixed for the critical dimension.

The approach that I will present here is closer in nature to the light-cone approach, in the sense that the quantization procedure will go through a complete reduction of the phase space of the Nambu-Goto string prior to quantization. Nevertheless, and in contrast to the light-cone approach, it will be possible to keep explicitly Poincaré, or rather Euclidean, invariance all along the way. The key ingredient to the construction will be the Enneper-Weierstrass representation of minimal surfaces. As I will show, following otherwise completely standard geometrical constructions, the reduced phase space of the three-dimensional closed Nambu-Goto string can be locally identified with the space of complex analytic functions plus three translational zero modes. In the reduction process the conformal structure is completely fixed by choosing a geometrical parameterization of the surface in terms of its Gauss map by stereographic projection. As a consequence of this, rotational invariance is explicitly implemented as a  $SU(2)$  subgroup of the standard linear fractional transformations acting on the Riemann sphere, and realized linearly on the physical fields.

Of course, quantization requires something more than the identification of the reduced phase space, an explicit expression for the induced symplectic structure is mandatory. Surprisingly enough this will prove to be a simple task. The required machinery is provided by the covariant phase space approach to hamiltonian mechanics together with some basic symplectic geometry. Even though we will be working with infinite-dimensional manifolds, an adequate algebraization of the required geometrical constructions will allow us to carry out the reduction process in a rather standard fashion.

The plan of the paper is as follows. First I will remind the reader of some basic notions about the geometry of immersed surfaces in  $\mathbb{R}^3$  that will be used in the following. Then I will introduce the Enneper-Weierstrass representation of minimal surfaces, and I will show how Euclidean invariance is realized within this representation.

Next, I will briefly recall some general results about the three-dimensional

Nambu-Goto string and how they fit in the context of minimal surface theory. After this, and in order to understand how to define a symplectic structure in the reduced phase space of the theory, I will digress on the covariant phase space approach to classical mechanics, and I will explicitly show how to apply it in this particular case.

Finally, I will attack the quantization of the model. I will show, that in contrast to the standard approach, the quantization in this case yields as a result a target-space Euclidean-invariant, positive-definite, two-dimensional quantum field theory. The associated physical states, which are obtained by the repeated action of creation operators on the vacuum state, can then be identified, via the Wigner method of induced representations applied to the Euclidean group, with particle states of integer spin and arbitrary mass.

## 2 A very brief course about surface theory in $\mathbb{R}^3$

The purpose of this introductory section is to present in a simple manner the most important geometrical constructions to be used in the sequel, as well as to set up my notations. For a comprehensive introduction to this fascinating subject I refer the reader to the excellent book of M. Spivak [9].

Let  $\Sigma$  be an oriented two-dimensional connected Riemannian manifold and  $\mathbf{X} : \Sigma \rightarrow \mathbb{R}^3$  an isometric immersion of  $\Sigma$  into  $\mathbb{R}^3$ . At any point  $p$  of  $\Sigma$  a basis for the tangent plane is provided by  $\partial_\alpha X^i$ . The induced metric, or first fundamental form of the immersion, is then given by

$$g_{\alpha\beta} = \partial_\alpha \mathbf{X} \cdot \partial_\beta \mathbf{X}. \quad (1)$$

It is now possible to obtain a basis for  $T\mathbb{R}^3$  at  $p$  by adding a unitary perpendicular vector  $\mathbf{n}$ , whose explicit coordinate expression may be given by

$$n^i = \frac{1}{2\sqrt{g}} \epsilon^{ijk} \epsilon^{\alpha\beta} \partial_\alpha X^j \partial_\beta X^k, \quad (2)$$

with  $g$  being the determinant of the induced metric.

One may now write down the structural equations of the immersion as

$$\partial_\beta \partial_\alpha \mathbf{X} = \Gamma_{\beta\alpha}^\rho \partial_\rho \mathbf{X} + K_{\beta\alpha} \mathbf{n} \quad (3)$$

$$\partial_\alpha \mathbf{n} = -g^{\beta\rho} K_{\alpha\beta} \partial_\rho \mathbf{X}. \quad (4)$$

The first of these equations may be taken as the definition of the extrinsic curvature  $K$ , or second fundamental form of the immersion, while the second follows from consistency with the relations  $\mathbf{n} \cdot \mathbf{n} = 1$  and  $\partial_\alpha \mathbf{X} \cdot \mathbf{n} = 0$ . Notice that multiplying the first of these equations by  $\partial_\gamma \mathbf{X}$  one readily obtains that the connection coefficients  $\Gamma$  are the ones of the Levi-Civita connection associated with the induced metric; multiplication by  $\mathbf{n}$  implies that  $K$  is a symmetric tensor.

The Codazzi-Mainardi equation is obtained from

$$\partial_\gamma \mathbf{X} \cdot (\epsilon^{\alpha\beta} \partial_\alpha \partial_\beta \mathbf{n}) = 0, \quad (5)$$

which yields that  $\nabla_{[\alpha} K_{\beta]\gamma} = 0$ . And finally the Gauss equation is obtained from

$$\partial_\gamma \mathbf{X} \cdot (\epsilon^{\rho\beta} \partial_\rho \partial_\beta \partial_\alpha \mathbf{X}) = 0, \quad (6)$$

which implies that  $R_{\gamma\alpha\rho\beta} = K_{\gamma[\rho} K_{\beta]\alpha}$ , where  $R$  is the Riemann curvature tensor associated with the induced metric.

It is now intuitively clear that given two symmetric tensors  $g$  and  $K$  obeying the integrability condition one may recover, up to Euclidean motions<sup>2</sup>, the associated surface by integrating the structural equations.

One may define now the mean curvature,  $H$ , and the Gaussian curvature,  $K$ , by

$$H = \frac{1}{2} g^{\alpha\beta} K_{\alpha\beta}, \quad \text{and} \quad K = \frac{1}{2} \epsilon^{\alpha\rho} \epsilon^{\beta\gamma} K_{\alpha\beta} K_{\rho\gamma}. \quad (7)$$

### 2.1 The Enneper-Weierstrass representation of minimal surfaces

Minimal surfaces are defined via the condition  $H = 0$ . They owe their name to the fact that they minimize the area functional, and therefore are solutions to the Nambu-Goto variational problem.

It will be useful in the following to introduce an isothermal coordinate system. Isothermal coordinates are defined through the condition that the induced metric is proportional to the standard flat two-dimensional Euclidean metric. That such a coordinate system can always be locally achieved is a standard result

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<sup>2</sup> This is due to the fact that the first and second fundamental forms, as defined above, are invariant under global translations and rotations in  $\mathbb{R}^3$ .

which proof can be found, for example, in [9]. If we denote such a coordinate system by  $(x, y)$  the zero mean curvature condition reads

$$\frac{\partial^2 \mathbf{X}}{\partial x^2} + \frac{\partial^2 \mathbf{X}}{\partial y^2} = 0, \quad (8)$$

which we can also write as

$$\frac{\partial}{\partial x} \left( \frac{\partial \mathbf{X}}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\partial \mathbf{X}}{\partial y} \right). \quad (9)$$

Therefore there is locally a vectorial function  $\mathbf{Y}$  such that

$$\frac{\partial \mathbf{Y}}{\partial x} = -\frac{\partial \mathbf{X}}{\partial y} \quad \text{and} \quad \frac{\partial \mathbf{Y}}{\partial y} = \frac{\partial \mathbf{X}}{\partial x}. \quad (10)$$

But these are none other than the Cauchy-Riemann equations for  $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$ . Therefore the minimality condition implies that  $\mathbf{X}$  is the real part of a complex analytic function  $\mathbf{Z}$ . Notice, however, that the condition that  $(x, y)$  constitute an isothermal coordinate function yields a further constraint in  $\mathbf{Z}$ , i.e.,

$$\frac{\partial \mathbf{Z}}{\partial z} \cdot \frac{\partial \mathbf{Z}}{\partial z} = 0, \quad (11)$$

where  $z = x + iy$ . Expanding in components the above expression one directly gets the standard conditions

$$\frac{\partial \mathbf{X}}{\partial x} \cdot \frac{\partial \mathbf{X}}{\partial x} = \frac{\partial \mathbf{X}}{\partial y} \cdot \frac{\partial \mathbf{X}}{\partial y} \quad \text{and} \quad \frac{\partial \mathbf{X}}{\partial x} \cdot \frac{\partial \mathbf{X}}{\partial y} = 0. \quad (12)$$

It is therefore natural to introduce the complex analytic function  $\boldsymbol{\psi} = \partial \mathbf{Z}$ , where  $\partial$  stands as a shorthand for  $\partial/\partial z$ . Then one can naturally associate to every minimal surface a quadric in  $\mathbb{C}^3$  defined through  $\boldsymbol{\psi} \cdot \boldsymbol{\psi} = 0$ , and viceversa via

$$\mathbf{X}(z, \bar{z}) = \text{Re} \int^z \boldsymbol{\psi}(\omega) d\omega, \quad (13)$$

up to an arbitrary translation.

The Enneper-Weierstrass representation is now achieved by finding an explicit solution for the quadratic equation in  $\boldsymbol{\psi}$ . It is straightforward to check that

$$\psi_1 = \frac{1}{2}f(1 - g^2), \quad \psi_2 = \frac{i}{2}f(1 + g^2), \quad \psi_3 = fg, \quad (14)$$

with  $g$  a meromorphic function and  $f$  complex analytic, and such that it has a zero of order  $2n$  wherever  $g$  has a pole of order  $n$ , is a solution of  $\psi \cdot \psi = 0$ . The converse is equally simple to prove. Notice that the quadratic equation can be written as

$$(\psi_1 - i\psi_2)(\psi_1 + i\psi_2) = -\psi_3^2. \quad (15)$$

If  $\psi_3$  is the zero function then we choose  $g = 0$  and  $f = 2\psi_1$ . If  $\psi_3$  is not the zero function then  $\psi_1 - i\psi_2$  is also not the zero function and then one can define

$$f = \psi_1 - i\psi_2, \quad g = \frac{\psi_3}{\psi_1 - i\psi_2}, \quad (16)$$

with  $f$  analytic and  $g$  meromorphic. Moreover it follows that

$$\psi_1 + i\psi_2 = -\frac{\psi_3^2}{\psi_1 - i\psi_2} = -fg^2. \quad (17)$$

Thus the analyticity of left hand side of the above equation implies the condition on the zeros and poles of  $f$  and  $g$  to be the one stated above. So we finally arrive to the Enneper-Weierstrass representation of minimal surfaces. Explicitly

$$X^1(z, \bar{z}) = \operatorname{Re} \int^z \frac{1}{2} f(\omega)(1 - g(\omega)^2) + x^1, \quad (18)$$

$$X^2(z, \bar{z}) = \operatorname{Re} \int^z \frac{i}{2} f(\omega)(1 + g(\omega)^2) + x^2, \quad (19)$$

$$X^3(z, \bar{z}) = \operatorname{Re} \int^z f(\omega)g(\omega) + x^3, \quad (20)$$

with the  $x^i$  real constants.

One can do still better, however, as I will now go on to show. Before doing so one should introduce a further geometrical construction: the Gauss map. The Gauss map is given by associating to each point in the surface its unit normal vector  $\mathbf{n}$ . This, of course, requires the choice of an orientation, therefore we will restrict from now on our considerations to orientable surfaces. A direct computation yields

$$\mathbf{n} = \left( \frac{2\operatorname{Re} g(z)}{|g(z)|^2 + 1}, \frac{2\operatorname{Im} g(z)}{|g(z)|^2 + 1}, \frac{|g(z)|^2 - 1}{|g(z)|^2 + 1} \right) \in S^2. \quad (21)$$

Notice that as we approach a pole of  $g(z)$   $\mathbf{n} \rightarrow (0, 0, 1)$ .

If we are not at a flat point of the surface it is possible to fix the conformal structure by choosing as our local coordinate the image under stereographic projection of the two-sphere onto the complex plane plus the point at infinity. This is equivalent to fixing  $g(\omega) = \omega$ . Therefore one may write

$$\mathbf{X}(z, \bar{z}) = \frac{1}{2} \text{Re} \int^z f(\omega) (1 - \omega^2, i(1 + \omega^2), 2\omega) + \mathbf{x}. \quad (22)$$

One may now compute all the geometrically relevant quantities in terms of  $f$ . In particular

$$g_{z\bar{z}} = \frac{1}{2} f \bar{f} (1 + z\bar{z})^2. \quad (23)$$

More interesting for our interest will turn out to be the Hopf quadratic differential, also called skew curvature, which is nothing but the  $zz$  component of the extrinsic curvature. A direct computation yields

$$K_{zz} = \partial \bar{\partial} \mathbf{X} \cdot \mathbf{n} = f, \quad (24)$$

thus giving a direct geometrical interpretation to the analytic function  $f$ . As we will now see this result will show to be of the utmost importance. Notice also that the analyticity of the skew curvature is a direct consequence of the Codazzi-Mainardi equation restricted to minimal surfaces.

It will be of crucial importance for the string physics to understand how Euclidean invariance is realized in the Enneper-Weierstrass parameterization of minimal surfaces. Notice that this parameterization requires a soldering between target space degrees of freedom and world-sheet ones: we are using the stereographic projection of the unit normal vector to the surface to fix the conformal coordinates on the surface. It is therefore natural to expect that rotations in the target space are naturally associated with rotation of the Riemann sphere. Indeed from the formula above it may appear that  $f$  is invariant under rotations because all indices are properly contracted, nevertheless in order to have the surface parameterized through the Gauss map one should take a compensating transformation in the  $z$  coordinate. It is a simple exercise to check that rotations of the Riemann sphere correspond to an  $SU(2)$  subgroup of  $SL(2, \mathbb{C})$  given by

$$\tilde{\eta} = \frac{a\eta - b}{b\eta + \bar{a}} \quad \text{with} \quad a\bar{a} + b\bar{b} = 1. \quad (25)$$

Because  $f$  is a quadratic differential it should transform such that

$$\tilde{f}(\tilde{\eta}) d^2 \tilde{\eta} = f(\eta) d^2 \eta, \quad (26)$$



or explicitly for  $SU(2)$  transformations

$$\tilde{f}(\tilde{\eta}(\eta)) = (\bar{b}\eta + \bar{a})^4 f(\eta). \quad (27)$$

Let me check that the transformed  $f$  does indeed correspond to a rotated surface. For the time being I will ignore the zero modes that transform in the usual way. If one considers

$$\tilde{X}(z, \bar{z}) = \frac{1}{2} \text{Re} \int^z d\tilde{\eta} \tilde{f}(\tilde{\eta}) \begin{pmatrix} 1 - \tilde{\eta}^2 \\ i(1 + \tilde{\eta}^2) \\ 2\tilde{\eta} \end{pmatrix} \quad (28)$$

one may change variables in the integral to obtain

$$\tilde{X}(z, \bar{z}) = \frac{1}{2} \text{Re} \int^{\tilde{z}} d\eta f(\eta) \begin{pmatrix} (\bar{b}\eta + \bar{a})^2 - (a\eta - b)^2 \\ i(\bar{b}\eta + \bar{a})^2 + i(a\eta - b)^2 \\ 2(\bar{b}\eta + \bar{a})(a\eta - b) \end{pmatrix}, \quad (29)$$

with

$$\tilde{z} = \frac{\bar{a}z + b}{-\bar{b}z + a}. \quad (30)$$

If one chooses  $a = e^{i\theta/2}$  and  $b = 0$ , after a little algebra one arrives to the expression

$$\tilde{X} = \frac{1}{2} \text{Re} \int^{\tilde{z}} d\eta f(\eta) \begin{pmatrix} \cos \theta (1 - \eta^2) - i \sin \theta (1 + \eta^2) \\ \sin \theta (1 - \eta^2) + i \cos \theta (1 + \eta^2) \\ 2\eta \end{pmatrix}. \quad (31)$$

Therefore one finally obtains

$$\tilde{X}(z, \bar{z}) = \mathcal{R}(\mathbf{e}_3, \theta) X(\tilde{z}, \bar{\tilde{z}}), \quad (32)$$

with  $\mathcal{R}(\mathbf{e}_3, \theta)$  being the standard  $SO(3)$  matrix associated with a rotation of an angle  $\theta$  around the axis given by  $\mathbf{e}_3$ .

A little more of work shows that the case with  $a = \cos \phi/2$  and  $b = \sin \phi/2$  corresponds to a rotation of angle  $\phi$  around  $\mathbf{e}_2$ ; but as it is well-known, these

two rotations generate the whole  $SO(3)$  group of rotations, therefore showing the correctness of our assumption.

### 3 The Nambu-Goto action

Although this is by no means a review in string theory, the purpose of this section is to recall certain properties of the Nambu-Goto string that will be extensively used in the following, as well as to set up my notations and conventions.

The Nambu-Goto action is the simplest geometric invariant of an immersed surface, i.e., its area.

$$S(\Sigma) = -\frac{1}{4\pi\alpha} \text{Area}(\Sigma) \quad (33)$$

As already mentioned in the previous section, the solution of its associated variational problem is given by surfaces of zero mean curvature. In the conformal gauge, or equivalently isothermal coordinates, the solution takes the general form (I will work in a system of units in which  $\alpha = 1/2$ )

$$\mathbf{X}(z, \bar{z}) = \mathbf{x} - \frac{i}{4} \mathbf{p} \ln z \bar{z} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \boldsymbol{\alpha}_n z^{-n} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \bar{\boldsymbol{\alpha}}_n \bar{z}^{-n} \quad (34)$$

if one chooses periodic boundary conditions. But, as it is well known, extra constraints come from the reparameterization invariance of the action. They can be written in terms of  $\mathbf{X}$  as

$$\partial \mathbf{X} \cdot \partial \mathbf{X} = \bar{\partial} \mathbf{X} \cdot \bar{\partial} \mathbf{X} = 0. \quad (35)$$

I will restrict myself, from now on, to the closed string case. The geometrical reasons to do so is to avoid flat surfaces as solutions. Notice that the geometrical parameterization of the surface by its Gauss map requires to rule out that case, which is only a priori allowed for open string boundary conditions. With all of this in mind, one can now directly apply all the machinery developed in the previous chapter.

It will be convenient to split the field  $\mathbf{X}$  into its holomorphic and antiholomorphic parts as  $\mathbf{X}(z, \bar{z}) = \mathbf{X}(z) + \bar{\mathbf{X}}(\bar{z})$  with

$$\mathbf{X}(z) = \frac{1}{2} \mathbf{x} - \frac{i}{4} \mathbf{p} \ln z + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \boldsymbol{\alpha}_n z^{-n}, \quad (36)$$

$$\bar{\mathbf{X}}(\bar{z}) = \frac{1}{2}\mathbf{x} - \frac{i}{4}\mathbf{p}\ln \bar{z} + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n \bar{z}^{-n}. \quad (37)$$

In terms of these fields one may write the Enneper-Weierstrass map as

$$\mathbf{X}(z) = \frac{1}{2} \int^z f(\omega) \left(1 - \omega^2, i(1 + \omega^2), 2\omega\right) + \frac{1}{2}\mathbf{x}, \quad (38)$$

and the obvious equivalent expression for  $\bar{\mathbf{X}}(\bar{z})$ . From now on I will concentrate almost exclusively in the holomorphic part, the results being trivially extended to the antiholomorphic sector.

#### 4 The covariant phase space formalism

The original idea of developing the canonical formalism in an explicitly covariant manner is first due to Witten [4], and later developed by Crnkovic [6] and Zuckerman [5]. The idea is simple and is based on the observation that there is a one-to-one relationship among points in the phase space and solutions to the equation of motion. This is roughly equivalent to say that given initial conditions, that require a noncovariant choice of a space-time slice, the solution to the equations of motion is fully determined (of course, special care should be taken in the presence of gauge invariances). It is therefore natural in a covariant field theory to preserve explicitly the covariance properties of the theory and define the phase space directly as the solution space of the associated field theory.

Our final goal is to define Poisson brackets in the reduced phase space associated with the three-dimensional Nambu-Goto string. In order to do so I will have to define some basic objects: functions, vector fields and differential forms living in an infinite-dimensional phase space. To avoid getting into analytical details it will be useful to algebraize the necessary notions. This point of view may not be familiar to everyone, so it may be useful to introduce it firstly in the familiar setting of (finite-dimensional) classical hamiltonian dynamics.

Let me start with a simple example. Consider  $M = \mathbb{R}^{2n}$  for our phase space with coordinates  $(q^i, p_i)$ . The Poisson bracket of any two functions  $f$  and  $g$  is given by

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right). \quad (39)$$

It is easy to see that the Poisson bracket is antisymmetric and that it satisfies

the Jacobi identity. Therefore it gives the ring of functions on  $M$  the structure of a Lie algebra. More is true, however. The Poisson bracket is also easily seen to act as a derivation on the ring of functions: if  $f$ ,  $g$ , and  $h$  are functions on  $M$ ,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h. \quad (40)$$

These facts turn the functions on  $M$  into a Poisson algebra.

Suppose that we now change coordinates to  $x^i(q, p)$ . The fundamental Poisson bracket of these coordinates is given by

$$\Omega^{ij} = \{x^i, x^j\} = \sum_k \left( \frac{\partial x^i}{\partial q^k} \frac{\partial x^j}{\partial p_k} - \frac{\partial x^i}{\partial p_k} \frac{\partial x^j}{\partial q^k} \right). \quad (41)$$

It is easy to check that  $\Omega^{ij}$  transforms tensorially under an arbitrary change of coordinates and thus defines an antisymmetric bivector; That is, a rank 2 antisymmetric covariant tensor. Furthermore, one can check that  $\Omega^{ij}$  is non-degenerate so that its inverse  $\Omega_{ij}$  exists and defines a nondegenerate 2-form on  $M$ , called the symplectic form. The Jacobi identities of the Poisson bracket imply a differential relation on  $\Omega^{ij}$  which, when inverted, imply that the symplectic form is closed.

To summarize, starting with the usual coordinates  $(q, p)$  and the usual Poisson brackets, we have uncovered an underlying geometric structure: a non-degenerate closed 2-form. This may seem overkill for  $\mathbb{R}^{2n}$  but it allows us to define a Poisson structure on any manifold  $M$  possessing a symplectic form. Of course, around each point of  $M$ , we can choose coordinates  $(q, p)$  whose Poisson bracket are the standard ones. This is the essence of Darboux's theorem. It states that symplectic manifolds of the same dimension are locally isomorphic.

The Poisson bracket allows us to assign to every function  $f$  a hamiltonian vector field  $H_f$  as follows:

$$H_f \cdot g = \{f, g\}, \quad (42)$$

whose components in local coordinates are given by  $H_f^i = -\Omega^{ij} \partial_j f$ . Therefore the Poisson brackets can be directly written in terms of the symplectic form  $\Omega$  as

$$\{f, g\} = \Omega(H_f, H_g), \quad (43)$$

with

$$\Omega(H_f, \cdot) = -df \cdot, \quad (44)$$

as a direct computation in component reveals.

Because the symplectic form is a closed 2-form it is always possible, at least locally, to define a 1-form  $\theta$ , usually called the canonical 1-form, such that  $\Omega = d\theta$ . It is then possible to define the Poisson brackets directly in terms of the canonical 1-form with the help of the Lie formula, i.e.,

$$\{f, g\} = d\theta(H_f, H_g) = H_f \cdot \theta(H_g) - H_g \cdot \theta(H_f) - \theta([H_f, H_g]). \quad (45)$$

We will still need another piece of geometrical information regarding presymplectic rather than symplectic manifolds. It is usual in dynamical systems to work with coordinates in phase space that are subject to constraints. Although physicists are well acquainted with Dirac's treatment of constraints in Lagrangian systems, for the case at hand we will need a more geometrical, though equivalent, approach to the subject. Fortunately, the ideas and techniques involved are still simple enough to be presented succinctly.

I will concentrate in the reduction to a submanifold of the original phase space defined by second class constraints; a symplectic submanifold in the standard geometrical nomenclature [3]. In this particular case the reduction process is trivial: if we denote by  $C$  the symplectic submanifold, one can naturally define a symplectic form on it simply by restriction of the original 2-form  $\Omega$ , which is usually denoted by  $\Omega|_C$ . More precisely, if  $Z$  and  $Y$  belong to the tangent space at a point  $p$  of  $C$

$$\Omega|_C(Z, Y) = \Omega(Z, Y), \quad (46)$$

where in the right hand side  $Z$  and  $Y$  are considered as elements of  $T_p M$ . It is somehow more tedious, although straightforward, to show that this natural geometric construction corresponds to the standard Dirac bracket prescription. The interested reader can find a proof of this fact in [10].

It will be more convenient to think of the reduced phase space in a more intrinsic way and look at  $C$  as the embedding of the physical phase space  $M_0$  into  $M$ . If we denote the embedding map by  $\varphi$  one can define a symplectic 2-form  $\omega$  in  $M_0$  acting on any two vector fields  $Z, Y \in TM_0$  by

$$\omega(Z, Y) = \varphi^* \Omega(Z, Y) = \Omega(\varphi_* Z, \varphi_* Y), \quad (47)$$

where  $\varphi_* Z$  stands for the pushforward of the vector field  $Z$  in  $M_0$  through the

map  $\varphi$ .

With all of this in mind we can now go on to analyze the case at hand. Rather than describing the general formalism that allows the construction of symplectic structures in the solution space of a dynamical system, I will work in detail the example in which we are interested at present.

The solution space associated with the Nambu-Goto equation of motions can be described as the solution space of the equation

$$\partial\bar{\partial}\mathbf{X} = 0, \quad (48)$$

with adequate boundary conditions, more on this will follow, and subject to the constraints

$$\partial\mathbf{X} \cdot \partial\mathbf{X} = \bar{\partial}\mathbf{X} \cdot \bar{\partial}\mathbf{X} = 0. \quad (49)$$

In order to construct the symplectic structure I will first concentrate in the unreduced manifold  $M$ , i.e., solutions of the equations  $\bar{\partial}\partial\mathbf{X}$ . Like in the finite-dimensional case one should first identify the functions, vector fields and one-forms in  $M$ . The functions, or rather functionals, are simply maps from the space  $M$  into the reals. A canonical example is supplied by the evaluation map, that is obtained by choosing a space-time point and evaluating a solution of the equation of motion in that point. We will be particularly interested in linear functionals. An example of them is supplied by the weighted integral in  $\mathbb{R}^3$  of a solution to the equation of motion, i.e.,

$$F_{\boldsymbol{\eta}}(\mathbf{X}) = \int_{\Sigma} \boldsymbol{\eta} \cdot \mathbf{X}, \quad (50)$$

although other natural examples will pop up in the following.

Vector fields are naturally parameterized by deformations of the solutions preserving the equations of motion, i.e., symmetries. In our case, since the equations defining  $M$  are linear, vector fields are themselves parameterized by solutions of the equation itself. Their action on functions is defined as usual by

$$\partial_{\mathbf{A}}G(\mathbf{X}) = \frac{d}{d\epsilon}G(\mathbf{X} + \epsilon\mathbf{A})|_{\epsilon=0}, \quad (51)$$

where  $\bar{\partial}\partial\mathbf{A} = 0$ .

Finally we should define the gradient of a function  $G$  or its associated 1-form  $dG$ . One can easily do so by defining its action on vector fields to be

$$dG(\partial_A) = \partial_A G, \quad (52)$$

mimicking the standard finite-dimensional definition. From now on, and in order not to clutter up the notation, I will confuse the vector fields with the solutions of the equation of motion parameterizing them. It will be convenient to center our attention in the holomorphic sector. It will be useful to explicitly split the zero modes and write an holomorphic deformation  $A(z)$  as

$$A(z) = \frac{1}{2}a - \frac{i}{4}A_0 \ln z + A_z(z), \quad (53)$$

with

$$A_z = \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} A_n z^{-n}. \quad (54)$$

Let me show now that the following canonical 1-form  $\theta$  in  $M$

$$\theta(A) = \frac{1}{\pi} \oint dz \partial X \cdot A_z + \mathbf{p} \cdot \mathbf{a}, \quad (55)$$

does the required job, where  $\oint$  stands for the Cauchy integral for a contour that has  $z = 0$  as an interior point, and we are considering closed string boundary conditions for  $X$  and consequently for  $A$ . One should now compute, with the use of the Lie formula, the explicit expression for the symplectic form  $\omega$ , but rather than writing the general expression I will restrict myself to the case of constant vector fields in the covariant phase space, i.e., vector fields that are field independent. For that particular case a direct computation yields

$$\Omega(A, B) = \frac{2}{\pi} \oint dz \partial A_z \cdot B_z + (A_0 \cdot \mathbf{b} - \mathbf{a} \cdot B_0). \quad (56)$$

It is now simple to check that the 2-form defined above induces the usual Poisson brackets among the modes of the string. If one considers the linear functions

$$F_p^i = \frac{1}{\pi} \oint dz z^p \partial X^i, \quad (57)$$

one directly obtains that  $F_p^i = \alpha_p^i$  for  $p \neq 0$ . Therefore

$$\{F_p^i, F_r^j\} = \{\alpha_p^i, \alpha_r^j\}. \quad (58)$$

In order to compute this Poisson brackets one should first compute the hamiltonian vector fields associated with these linear functionals. For example,  $H_{F_p^i}$  it is obtained from

$$\Omega(H_{F_p^i}, B) = -dF_p^i(B) = -\frac{1}{\pi} \oint z^p \partial B^i, \quad (59)$$

which, together with the definition of  $\Omega$ , automatically implies that

$$H_{F_p^i} = \frac{1}{2} z^p \mathbf{e}_i, \quad (60)$$

where the  $\mathbf{e}_i$ 's form the standard basis for vectors in  $\mathbb{R}^3$ . From here directly follows that

$$\{\alpha_p^i, \alpha_r^j\} = dF_r^j(H_{F_p^i}) = \frac{1}{2\pi} \oint z^r \partial z^p \delta_{ij} = ip \delta^{ij} \delta_{p+r,0}. \quad (61)$$

A similar computation yields the lacking Poisson bracket

$$\{p^i, x^j\} = \delta^{ij}, \quad (62)$$

with all the remaining Poisson brackets being zero. Notice that under analytic continuation to Minkowski space, with our conventions,  $p^3$  and  $x^3$  pick up an extra factor of  $i$  and therefore one obtains the standard Minkowskian Poisson brackets  $\{p^i, x^j\} = \eta^{ij}$ . Thus proving the equivalence of the covariant phase space approach and standard symplectic methods for the case at hand.

#### 4.1 The Poisson brackets in the reduced phase space

Of course, this long detour has not been a caprice and I will go on now to show how the previous developments will allow us to compute the Poisson brackets in the reduced phase space. We are interested in computing the fundamental induced Poisson brackets on the modes of  $f(z)$ . If we define

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{f_n}{z^{n+2}}, \quad (63)$$

we will be naturally interested in the following functions

$$G_p = \frac{1}{2\pi i} \oint dz z^{p+1} f(z). \quad (64)$$



Let me first consider the case for which  $p \neq 0, \pm 1$ . As before one should start by finding the explicit expression of the hamiltonian vector field associated with  $G_p$ . From the definitions one directly gets

$$\omega(H_{G_p}, \xi) = \varphi^* \Omega(H_{G_p}, \xi) = -dG_p(\xi) = -\frac{1}{2\pi i} \oint dz z^{p+1} \xi(z). \quad (65)$$

The definition of the Enneper-Weierstrass map together with the 2-form  $\Omega$  defined above yield

$$\varphi^* \Omega(H_{G_p}, \xi) = \frac{1}{\pi} \oint dz \xi(z) \int^z d\eta H_{G_p}(\eta) (z - \eta)^2. \quad (66)$$

From where one obtains that

$$H_{G_p} = \frac{i}{4} p(p^2 - 1) z^{p-2}. \quad (67)$$

Therefore from this follows that

$$\{G_p, G_q\} = \{f_p, f_q\} = H_{G_p} \cdot G_q = \frac{p(p^2 - 1)}{8\pi} \oint dz z^{q+1} z^{p-2}, \quad (68)$$

or

$$\{f_p, f_q\} = \frac{i}{4} p(p^2 - 1) \delta_{p+q, 0} \quad (69)$$

for  $p \neq 0, \pm 1$ .

In order to compute the remaining Poisson brackets it will show convenient to introduce the following notation:

$$p^1 = 2i(f_1 - f_{-1}), \quad p^2 = -2(f_1 + f_{-1}), \quad \text{and} \quad p^3 = 4if_0. \quad (70)$$

It is now an straightforward computation to show that

$$\{p^i, x^j\} = \delta^{ij}, \quad (71)$$

with all other remaining Poisson brackets being zero. Notice that for the momentum associated with the zero modes the Enneper-Weierstrass is one to one, the above result directly reflects that fact, as can be directly checked from the definition of  $X$  in terms of the  $f_j$ .

## 5 The quantum theory

Now we have all the required information to construct the associated quantum theory. The first step is trivial and correspond to the standard rule of substituting Poisson brackets for commutators, i.e.,  $[ , ] = -i\{ , \}$ . In our case the Heisenberg quantization rule ( $\hbar = 1$ ) yields

$$[\hat{f}_p, \hat{f}_q] = \frac{1}{4}p(p^2 - 1)\delta_{p+q,0}, \quad (72)$$

and an identical expression for the modes of  $\hat{\bar{f}}_n$ , together with

$$[\hat{x}^i, \hat{p}^j] = i\delta^{ij}. \quad (73)$$

Notice that the mode algebra of the  $\hat{f}_n$  corresponds to the central term of the Virasoro algebra for  $D = 3$ . Although, at this point, I do not know if this is something more than sheer coincidence.

The ‘in’ vacuum state  $|0\rangle$  is defined trough the conditions

$$\hat{f}_n|0\rangle = 0 \quad \text{for } n \geq -1. \quad (74)$$

All states of momentum  $p$  are obtained from the action of creation operators, i.e.,  $\hat{f}_n$  with  $n \leq -2$ , acting on the state  $|0, p\rangle$  which is defined to be annihilated by all  $\hat{f}_n$  with  $n \geq 2$  and such that

$$\hat{p}^i|0, p\rangle = p^i|0, p\rangle. \quad (75)$$

The ‘out’ vacuum state  $\langle 0|$  is equally easily defined by taking the adjoint of  $\hat{f}_n$  to be

$$\hat{f}_n^\dagger = \hat{f}_{-n}, \quad (76)$$

which corresponds with

$$\hat{f}^\dagger(z) = \frac{1}{\bar{z}^4}\hat{f}\left(\frac{1}{\bar{z}}\right). \quad (77)$$

The reader may worry about the fact that although this rule seems natural from the conformal field theory point of view it may be unwarranted in this case, where the conformal invariance has been explicitly broken. But once again the geometry of surfaces comes to our rescue. In fact it is a simple

exercise to check that the surface defined through an  $\bar{f}(\bar{z})$  given by  $f^\dagger$ , as defined above, is the same as the one obtained from the original  $f$  up to a change in parameterization given by  $z \rightarrow 1/\bar{z}$  plus a reflection in the  $X^3$  coordinate. But, let me recall, that from the Euclidean field theory point of view the adjoint operation requires a time inversion due to the lack of a factor of  $i$  in the evolution operator. In radial quantization this is equivalent to the transformation  $z \rightarrow 1/\bar{z}$  [2]. The extra sign factor in the third coordinate can be equally understood in terms of the extra factor of  $i$  required to pass from Euclidean to Minkowskian signature in the target space-time. One may equally check that this hermitian conjugation rule comes naturally imposed from the usual one of the string modes in Minkowski space-time.

Notice also that this definition of adjoint imply that the momentum operators  $p^1, p^2$ , and  $ip^3$ , as defined from equation (70), are selfadjoint.

More importantly the above adjoint properties of the operator  $\hat{f}$  imply that the inner product of its associated Hilbert spaces, i.e., the ones obtained by the repeated application of creation operators onto the reference state  $|0, p\rangle$ , is positive-definite. A crucial property of the quantum theory required for consistency with the standard probabilistic interpretation of the transition amplitudes.

### 5.1 Quantum realization of Euclidean invariance

We have already studied in previous sections how rotational invariance was implemented at the classical level within the Enneper-Weierstrass representation of minimal surfaces. In our case, and in contrast with other gauge fixings like the light-cone one, the implementation of the Euclidean invariance<sup>3</sup> at the quantum level is completely straightforward. In fact it is a simple computation to show that the following operators

$$J^3 = -4 \sum_{n \geq 2} \frac{1}{n^2 - 1} : \hat{f}_{-n} \hat{f}_n : \quad (78)$$

$$J^2 = 2i \sum_{n \geq 2} \frac{1}{n(n+1)} : \hat{f}_{-n-1} \hat{f}_n : - 2i \sum_{n \geq 2} \frac{1}{n(n-1)} : \hat{f}_{-n+1} \hat{f}_n : \quad (79)$$

$$J^1 = -2 \sum_{n \geq 2} \frac{1}{n(n+1)} : \hat{f}_{-n-1} \hat{f}_n : - 2 \sum_{n \geq 2} \frac{1}{n(n-1)} : \hat{f}_{-n+1} \hat{f}_n : \quad (80)$$

obey the standard  $SU(2)$  algebra  $[J^i, J^j] = i\epsilon^{ijk} J^k$ , and moreover generate the right transformations on the modes of  $\hat{f}$ . Of course, the  $x^j$  and  $p^j$  operators

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<sup>3</sup> Translational symmetry is simply implemented by shifts in the zero modes.

transform in the standard fashion, and I will omit the complete expression of the  $J$ 's involving them and the modes of the operator associated with  $\bar{f}$ .

It is also instructive to notice that Euclidean invariance of the two point correlation function for the  $\hat{f}$  operator requires that

$$\langle \hat{f}(z) \hat{f}(0) \rangle = \frac{c/2}{z^4}. \quad (81)$$

That is, of course, the result obtained in this case with  $c = 3$ .

All of this completes the proof that the Nambu-Goto string when quantized in the reduced covariant phase space approach yields a consistent and Euclidean-invariant, two-dimensional quantum field theory.

## 5.2 Particle states and spectrum

The physical states are obtained by the repeated action of the creation modes of the fields  $f$  and  $\bar{f}$  on the momentum state  $|0, p\rangle$ . They will be labeled by two multi-indices as follows

$$|\{j, \bar{j}\}, p\rangle = \hat{f}_{-j_1} \cdots \hat{f}_{-j_n} \hat{\bar{f}}_{-\bar{j}_1} \cdots \hat{\bar{f}}_{-\bar{j}_n} |0, p\rangle \quad (82)$$

It is simple to show now that they carry irreducible representations of the Euclidean group associated with particles of integer spin and arbitrary mass. In order to do so, it will be convenient to recall some basic facts about Wigner method of induced representations applied to the three-dimensional Euclidean group.

One should start by identifying the Casimir operators. They are given by  $p^2$  and  $S = p \cdot J$ . I will denote by  $m^2$  and  $sm$  its respective eigenvalues, with  $m$  the mass of the particle state and  $s$  its spin. Therefore our particle states will be labeled by its mass and spin.

The components of the momentum  $p^j$  commute among themselves, so it is natural to label the physical states in terms of their eigenvectors. It is now standard to obtain the irreducible representations of the Euclidean group by choosing a standard momentum  $k^j$  and inducing the representations of the full Euclidean group from those of the stability subgroup associated to that particular momentum  $k^j$ , i.e., its little group. Let me reproduce the standard procedure for the case when  $k = (0, 0, m)$ , although, of course, any other momentum will do. The little group leaving invariant this reference momentum is one-dimensional and its algebra is generated by  $J^3$ . Its irreducible representations are given by a phase  $e^{-is\omega}$ , where  $\omega$  is the angle of rotation. The action

of any rotation  $\Lambda$  on any state  $|p, s\rangle$  can be obtained in an standard fashion as follows. Let me denote by  $R$  the rotation such that

$$p^j = R_i^j(p)k^i. \quad (83)$$

We can then define the state  $|p, s\rangle$  by

$$|p, s\rangle \sim U(R(p))|k, s\rangle, \quad (84)$$

up to an unimportant normalization factor.

The action of an arbitrary rotation  $\Lambda$  may be now easily computed as follows.

$$U(\Lambda)|p, s\rangle \sim U(\Lambda R(p))|k, s\rangle \sim U(R(\Lambda p))U(R^{-1}(\Lambda p)\Lambda R(p))|k, s\rangle. \quad (85)$$

The rationale behind this last step being that  $U(R^{-1}(\Lambda p)\Lambda R(p))$  leaves invariant the momentum  $k$ , i.e., it belongs to its little group. As commented above its irreducible representations are one-dimensional and one obtains

$$U(\Lambda)|p, s\rangle \sim e^{-is\omega_\Lambda}|\Lambda p, s\rangle, \quad (86)$$

where  $\omega_{\Lambda,p}$  is the Wigner rotation angle, and can be directly computed from the above expressions.

From all of this and the definition of  $J^3$  it now follows that the state  $|\{j, \bar{j}\}, p\rangle$  corresponds to a irreducible representation of the Euclidean group of mass  $p^2$  and spin

$$s = \sum_{\{j\}} j_i - \sum_{\{\bar{j}\}} \bar{j}_i. \quad (87)$$

Notice that due to the fact that the constraints have been solved there is no, at least to the best of my understanding, quantization condition on the mass spectrum in this scheme. Therefore the particle spectrum consists of particles of integer spin and arbitrary mass.

As a final comment, notice that in the quantization process I have assumed that the Enneper-Weierstrass parameterization is globally defined, which clearly implies that the surfaces is of genus zero, i.e., the surfaces result from a map of the Riemann sphere into spacetime. Therefore, in physicists' language, I have restricted myself to the free case. There is already a vast literature on how to extend the operator formalism to attack the interacting, or higher genus, theory. The application of those methods to this particular case will be a problem for the (near?) future.

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